

# Long-time behavior of stochastic model with multi-particle synchronization

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## Abstract

We consider a basic stochastic particle system consisting of  $N$  identical particles with isotropic  $k$ -particle synchronization,  $k \geq 2$ . In the limit when both number of particles  $N$  and time  $t = t(N)$  grow to infinity we study an asymptotic behavior of a coordinate spread of the particle system. We describe three time stages of  $t(N)$  for which a qualitative behavior of the system is completely different. Moreover, we discuss the case when a spread of the initial configuration depends on  $N$  and increases to infinity as  $N \rightarrow \infty$ .

**1. Introduction.** A wide class of probabilistic models can be interpreted as stochastic particle systems with a synchronization-like interaction. In general terms, the matter concerns a special class of jump Markov processes  $x(t) = (x_1(t), \dots, x_N(t))$  evolving in continuous time and taking their values in  $\mathbb{R}^N$ , with generators of the following symbolic form  $L = L_0 + L_s$ . The variable  $x_i \in \mathbb{R}$  is interpreted as a coordinate of the particle  $i$ . A free dynamics  $L_0$  corresponds to independent movements of the individual particles, and  $L_s$  corresponds to synchronizing jumps  $x = (x_1, \dots, x_N) \rightarrow x' = (x'_1, \dots, x'_N)$  which, according to many preceding papers [2–4, 7], fit to the following general rule

$$\{x'_1, \dots, x'_N\} \subset \{x_1, \dots, x_N\}, \quad \{x'_1, \dots, x'_N\} \neq \{x_1, \dots, x_N\}.$$

First mathematical papers on stochastic synchronization dealt with the case  $N = 2$  (see, for example, [1]). Many-particle systems ( $N > 2$ ), studied till now, have different forms of the synchronizing interaction. So [2–4] considered *pairwise* interactions, while in [7] a *three-particle* interaction was studied. In the present paper we consider a *general  $k$ -particle* symmetrized interaction which will be defined in terms of synchronizing maps.

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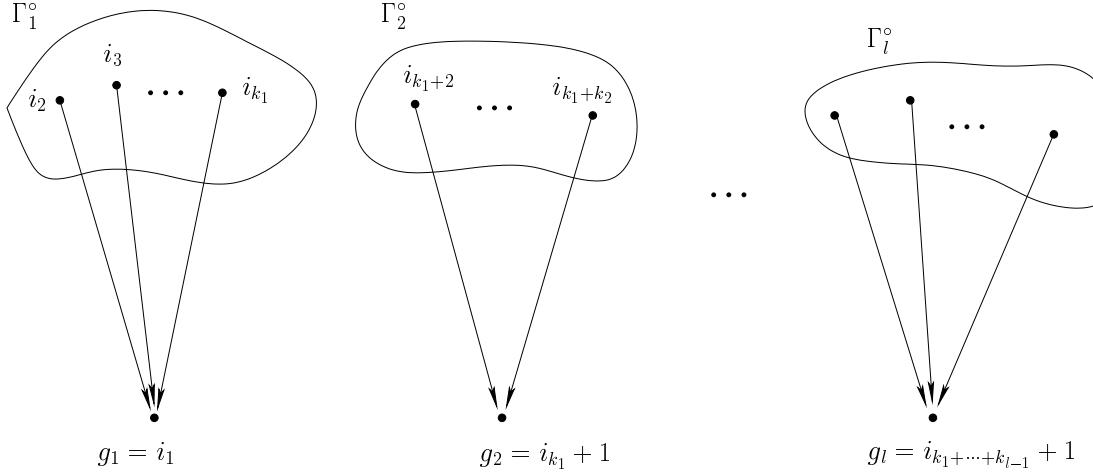


Figure 1: Correspondence between maps  $\pi_{k_1, \dots, k_l}$  and oriented graphs

When we analyze a collective behavior of such particle system we face with superposition of two opposite tendencies: with the course of time the free dynamics increases the spread of the particle system while the synchronizing interaction tries to decrease it. The aim of this paper is to study a qualitative balance between these two tendencies in the limit  $N \rightarrow \infty$ ,  $t = t(N) \rightarrow \infty$ . Our main result (Theorem 2) determines three phases of different behavior of the system depending on the growth rate of  $t(N)$ . Similar result was obtained in [4] for a system with *two types* of particles and a *pairwise* interaction between different types.

Main results of this paper were presented by the author at PTAP-2006 (see [8]).

**2. Synchronizing maps.** Let us enumerate the particles and let  $x_1, \dots, x_N$  be their coordinates. Let us fix a natural number  $k \geq 2$  and integers  $k_1 \geq 2, \dots, k_l \geq 2$  such that  $k_1 + \dots + k_l = k$ . We call the sequenced collection  $(k_1, \dots, k_l)$  a *signature* and will keep it fixed throughout the present paper. Let  $\mathcal{I}$  be a set of all sequenced collections  $(i_1, \dots, i_k)$ , consisting of  $k$  different elements of the index set  $\mathcal{N}_N := \{1, \dots, N\}$ . On the set  $\mathcal{I}$  we define a map  $\pi_{k_1, \dots, k_l} : (i_1, \dots, i_k) \mapsto (\Gamma_1, \dots, \Gamma_l)$ , where  $\Gamma_j = (g_j, \Gamma_j^\circ)$ ,

$$\begin{aligned}
 g_1 &= i_1, & \Gamma_1^\circ &= (i_2, \dots, i_{k_1}), \\
 g_2 &= i_{k_1+1}, & \Gamma_2^\circ &= (i_{k_1+2}, \dots, i_{k_1+k_2}), \\
 &\dots & & \\
 g_l &= i_{k_1+\dots+k_{l-1}+1}, & \Gamma_l^\circ &= (i_{k_1+\dots+k_{l-1}+1}, \dots, i_{k_1+\dots+k_l}).
 \end{aligned}$$

It is useful to associate with the map  $\pi_{k_1, \dots, k_l}$  an oriented graph as shown on the Figure 1.

Consider also a map  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ,

$$J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} : x = (x_1, \dots, x_N) \mapsto y = (y_1, \dots, y_N),$$

which is defined as follows:

$$y_m = \begin{cases} x_m, & \text{if } m \notin (i_1, \dots, i_k), \\ x_{g_j}, & \text{if } m \in (i_1, \dots, i_k), m \in \Gamma_j. \end{cases} \quad (1)$$

It is natural to call this map a *synchronization* of the collection of particles  $x_{i_1}, \dots, x_{i_k}$ , corresponding to the signature  $(k_1, \dots, k_l)$ . It is evident that the configuration  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} x$  has at least  $k_1$  particles with coordinates that are equal to  $x_{g_1}, \dots$ , at least  $k_l$  particles with coordinates that are equal to  $x_{g_l}$ .

**3. Multidimensional Markov process with synchronization.** We define a continuous time homogeneous Markov process  $x(t) = (x_1(t), \dots, x_N(t))$ ,  $t \geq 0$ , on the state space  $\mathbb{R}^N$  by means of the following formal generator  $L = L_0 + L_s$ . *Free dynamics*  $L_0$  corresponds to independent random walks and is chosen as

$$(L_0 f)(x) = \alpha \sum_{i=1}^N \int (f(x + z e_i) - f(x)) \rho(dz), \quad z \in \mathbb{R}, \quad e_i = (0, \dots, 1_i, \dots, 0),$$

i.e., independently of the other particles each particle  $i$  waits an exponential time with parameter  $\alpha > 0$  and performs a jump of the form  $x_i \rightarrow x_i + z$ , where  $z$  is distributed according to the law  $\rho(dz)$ , which is common for all particles. We assume that the distribution  $\rho$  has a compact support and is nontrivial in the following sense:  $b_2 := \int x^2 \rho(dx) > 0$ . Denote also  $a := \int x \rho(dx)$ .

*Synchronizing interaction*  $L_s$  has the form

$$L_s f = \frac{\delta}{N^{[k]}} \sum_{(i_1, \dots, i_k) \in \mathcal{I}} L_{s, (i_1, \dots, i_k)} f, \quad (2)$$

where  $N^{[k]} := N(N-1) \cdots (N-k+1)$  and

$$(L_{s, (i_1, \dots, i_k)} f)(x) = f(J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} x) - f(x), \quad x = (x_1, \dots, x_N), \quad (3)$$

for bounded continuous functions  $f$ . In other words, independently from the free dynamics at epochs of a Poissonian flow with parameter  $\delta > 0$  we choose with an equal probability one element  $(i_1, \dots, i_k)$  from the set  $\mathcal{I}$ , and synchronize the particle configuration  $x_1, \dots, x_N$  according to the map  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}$ . Hence, the interaction, corresponding to the given choice of  $(i_1, \dots, i_k)$ , consists in instantaneous and simultaneous transfer of particles of each group  $x_{k_1+\dots+k_{j-1}+1}, \dots, x_{k_1+\dots+k_j}$  to the point with the coordinate  $x_{g_j} = x_{k_1+\dots+k_{j-1}+1}$ ,  $j = \overline{1, l}$ .

Since in the sum (2) all summands have the same weight, the system under consideration belongs to the class of models with mean-field interaction.

**4. Main results.** Consider a function  $V : \mathbb{R}^N \rightarrow \mathbb{R}_+$

$$V(x) := \frac{1}{N(N-1)} \sum_{m < n} (x_m - x_n)^2.$$

It is easy to see that this function coincides with the *empiric (sample) variance*  $S^2$ , where

$$S^2 := \frac{1}{N-1} \sum_{m=1}^N (x_m - M(x))^2, \quad M(x) := \frac{1}{N} \sum_{m=1}^N x_m.$$

**Lemma 1**  $(LM)(x) = \alpha a$  for all  $x = (x_1, \dots, x_N)$ .

**Lemma 2** If  $N \rightarrow \infty$  then we have  $LV = \alpha b_2 - \frac{\delta \varkappa}{N(N-1)} V$  for some  $\varkappa > 0$ .

It follows from the proof that  $\varkappa = \sum_{j=1}^l k_j^2 - k$ .

We are interested in the following means:  $\mu_N(t) := \mathbb{E}M(x(t))$  and  $R_N(t) := \mathbb{E}V(x(t))$ . In the below statements we always assume that  $N \rightarrow \infty$ .

**Theorem 1** For any  $t > 0$   $\lim_N \frac{\mu_N(t) - \mu_N(0)}{t} = \alpha a$ . Moreover, for any function  $t(N) \rightarrow \infty$  the following convergence holds  $\frac{\mu_N(t(N)) - \mu_N(0)}{t(N)} \rightarrow \alpha a$ .

**Theorem 2** Assume that  $\sup_N |R_N(0)| < \infty$ . There exist three time scales  $t = t(N)$ , where  $R_N(t(N))$  has completely different asymptotic behavior.

- If  $\frac{t(N)}{N^2} \rightarrow 0$ , then  $R_N(t(N)) \sim \alpha b_2 t(N)$ .
- If  $t(N) = cN^2$ , then  $R_N(t(N)) \sim \alpha b_2 (\delta \varkappa)^{-1} (1 - \exp(-\delta \varkappa c)) N^2$ .
- If  $\frac{t(N)}{N^2} \rightarrow \infty$ , then  $R_N(t(N)) \sim \alpha b_2 (\delta \varkappa)^{-1} N^2$ .

In [4] similar consecutive stages were called correspondingly a phase of initial desynchronization, a phase of critical slowdown of desynchronization and a phase of final stabilization. We see that the first phase does not contribute to the asymptotic behavior.

**Theorem 3** Assume that  $R_N(0) \rightarrow \infty$  as  $N \rightarrow \infty$ . Then the additional condition  $\frac{R_N(0)}{t(N)} \rightarrow 0$  is sufficient to ensure the validity of the corresponding statements of Theorem 2.

**5. Proof of Lemma 1.** A straightforward calculation shows that  $(L_0 M)(x) \equiv \alpha a$ , hence we need only to prove that  $(L_s M)(x) \equiv 0$ . Let us show that this fact follows from the symmetry of the synchronizing interaction. Indeed, if the signature of interaction  $(k_1, \dots, k_l)$  is given we can fix some sets

$$B_j \in \mathcal{N}_N, \quad |B_j| = k_j, \quad (j = \overline{1, l}), \quad B_{j_1} \cap B_{j_2} = \emptyset \quad (j_1 \neq j_2)$$

and define  $\mathcal{I}_{k_1, \dots, k_l}^{B_1, \dots, B_l} := \left\{ (i_1, \dots, i_k) : \{g_j\} \cup \Gamma_j^\circ = B_j \quad \forall j = \overline{1, l} \right\}$ , where  $g_j$  and  $\Gamma_j^\circ$  are determined by the map  $\pi_{k_1, \dots, k_l}$  applied to  $(i_1, \dots, i_k)$ . To finish the proof it is sufficient to show that

$$\sum_{(i_1, \dots, i_k) \in \mathcal{I}_{k_1, \dots, k_l}^{B_1, \dots, B_l}} \left( M \circ J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} - M \right) = 0, \quad (4)$$

since  $\sum_{(i_1, \dots, i_k) \in \mathcal{I}} = \sum_{B_1, \dots, B_l} \sum_{(i_1, \dots, i_k) \in \mathcal{I}_{k_1, \dots, k_l}^{B_1, \dots, B_l}}$ . It is easy to see that

$$M \left( J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} x \right) - M(x) = \frac{1}{N} \sum_{j=1}^l \sum_{h \in \Gamma_j^\circ} (x_{g_j} - x_h). \quad (5)$$

Substituting the r.h.s. of (5) into the sum (4), we see that, since in (4)  $g_j$  covers the whole set  $B_j$ , for any pair of indices  $u, v \in B_j$  we have exactly one difference  $(x_u - x_v)$ , when  $g_j = u$ , and exactly one difference  $(x_v - x_u)$ , when  $g_j = v$ . Hence the sum (4) is equal to zero.

**6. Proof of Lemma 2.** Define a function  $f_{m,n} : \mathbb{R}^N \rightarrow \mathbb{R}$  by the formula

$$f_{m,n}(x) = \frac{1}{N(N-1)} (x_n - x_m)^2.$$

So  $V(x) = \sum_{m < n} f_{m,n}(x)$ . It is straightforward to check that  $L_0 f_{m,n}(x) = \left( N(N-1) \right)^{-1} 2ab_2$  and hence  $(L_0 V)(x) \equiv \alpha b_2$ .

We have

$$L_s V = \delta \sum_{m' < n'} \frac{1}{N^{[k]}} \sum_{(i_1, \dots, i_k) \in \mathcal{I}} L_{s, (i_1, \dots, i_k)} f_{m', n'}. \quad (6)$$

Consider a summand  $L_{s, (i_1, \dots, i_k)} f_{m', n'}$ . By (1) the map  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}$  transfers a particle with index  $m'$  to the point having the coordinate  $x_m$ , where

$$m = \begin{cases} m', & \text{if } m' \notin (i_1, \dots, i_k), \\ g_j, & \text{if } m' \in (i_1, \dots, i_k), m' \in \Gamma_j \end{cases} \quad (7)$$

Consequently,  $L_{s, (i_1, \dots, i_k)} f_{m', n'} = f_{m,n} - f_{m', n'}$  with some  $m$  and  $n$ , which are not necessarily

different from  $m'$  and  $n'$ . Hence, there exist such  $a_{nm}(N) \in \mathbb{R}$  that

$$(L_s V)(x) = \delta \sum_{m < n} a_{mn}(N) f_{m,n}(x). \quad (8)$$

Our goal is to show that the coefficients  $a_{mn}(N)$  do not depend on  $m$  and  $n$  and that

$$a_{mn}(N) = -\frac{\varkappa}{N(N-1)} \quad (9)$$

for some constant  $\varkappa > 0$ . Let us fix some pair  $\{m, n\}$  and calculate  $a_{mn}(N)$ . When we choose a collection  $(i_1, \dots, i_k)$  for the synchronization, in that way we choose  $g_1, \dots, g_l$  and  $\Gamma_1^\circ, \dots, \Gamma_l^\circ$ . Denote  $G = \{g_1, \dots, g_l\}$ . If the pair  $\{m, n\}$  is given then the set  $\mathcal{I}$  can be divided into four non-intersecting parts as follows:  $\mathcal{I} = \bigcup_{w=0}^3 \mathcal{I}_w^{m,n}$ , where

$$\begin{aligned} \mathcal{I}_0^{m,n} &:= \{(i_1, \dots, i_k) : \{m, n\} \cap (\cup_j \Gamma_j) = \emptyset\}, \\ \mathcal{I}_1^{m,n} &:= \{(i_1, \dots, i_k) : \{m, n\} \cap (\cup_j \Gamma_j^\circ) \neq \emptyset\}, \\ \mathcal{I}_2^{m,n} &:= \{(i_1, \dots, i_k) : \{m, n\} \subset G\}, \\ \mathcal{I}_3^{m,n} &:= \{(i_1, \dots, i_k) : |\{m, n\} \cap G| = 1\} \setminus \mathcal{I}_1^{m,n}. \end{aligned}$$

In each sum  $\sum^{(w)} := \frac{1}{N^{[k]}} \sum_{(i_1, \dots, i_k) \in \mathcal{I}_w^{m,n}} \sum_{m' < n'} \left( f_{m',n'} \circ J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} - f_{m',n'} \right)$  we pick out only summands that contain the function  $f_{m,n}$  and a coefficient in front of it will be denoted by  $a_{mn}^{(w)}(N)$ . By representations (6) and (8) we can write  $a_{nm}(N) = \sum_{w=0}^3 a_{mn}^{(w)}(N)$ .

0) If  $(i_1, \dots, i_k) \in \mathcal{I}_0^{m,n}$  then the particles with indices  $m$  and  $n$  are fixed points of the map  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}$ , hence  $a_{mn}^{(0)}(N) = 0$ .

I) If  $(i_1, \dots, i_k) \in \mathcal{I}_1^{m,n}$ , then the summand  $f_{m,n}$  can be presented in the sum  $\sum^{(1)}$  only with the sign “-” and only in the case when  $(m', n') = (m, n)$ . A total number of collections  $(i_1, \dots, i_k)$ , that belong to the set  $\mathcal{I}_1^{m,n}$ , is equal to  $|\mathcal{I}_1^{m,n}| = N^{[k]} - (N-2)^{[k-l]} (N-(k-l))^{[l]}$ . Therefore,

$$a_{mn}^{(1)}(N) = (-1) \frac{N^{[k]} - (N-2)^{[k-l]} (N-(k-l))^{[l]}}{N^{[k]}}.$$

II) For definiteness let us fix some  $i, j \in \{1, \dots, l\}$  and consider a subsum of  $\sum^{(2)}$  taken over such subset of  $\mathcal{I}_2^{m,n}$  that  $m = g_i, n = g_j$ . Under the action of the map  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)}$  each of function  $f_{uv}$ , where  $u \in \Gamma_i, v \in \Gamma_j$ , turns into the function  $f_{mn}$ . Thus in the above subsum we find  $k_i k_j$  summands  $f_{mn}$  with the sign “+” and only one summand (corresponding to the case  $(m', n') = (m, n)$ ) with

the sign “−”. Hence,

$$a_{mn}^{(2)}(N) = \sum_{i,j=1, i \neq j}^l (k_i k_j - 1) \frac{(N-2)^{[k-2]}}{N^{[k]}}.$$

III) Fix  $i \in \{1, \dots, l\}$  and consider in  $\sum^{(3)}$  a subsum taken over such  $(i_1, \dots, i_k)$  that  $m = g_i \in G$ ,  $n \notin G$ . A number of collections  $(i_1, \dots, i_k)$ , satisfying to this assumption, is equal to  $(N-2)^{[k-1]}$ . Under this assumption each of  $k_i$  functions  $f_{un}$ , where  $u \in \Gamma_i$ , turns into  $f_{mn}$ . Changing the roles of  $m$  and  $n$ , we conclude that

$$a_{mn}^{(3)}(N) = 2 \sum_{i=1}^l (k_i - 1) \frac{(N-2)^{[k-1]}}{N^{[k]}}.$$

As is easy to see the values of  $a_{mn}^{(1)}(N)$ ,  $a_{mn}^{(2)}(N)$  and  $a_{mn}^{(3)}(N)$  do not depend on  $m, n$ . After some calculations we obtain the following formulae

$$\begin{aligned} a_{mn}^{(1)}(N) &= -\frac{2(k-l)(N-k) + (k-l)(k+l-1)}{N(N-1)}, \\ a_{mn}^{(2)}(N) &= \frac{k^2 - l^2 + l - \sum_j k_j^2}{N(N-1)}, \\ a_{mn}^{(3)}(N) &= \frac{2(k-l)(N-k)}{N(N-1)}. \end{aligned}$$

Summing these values together, we get

$$a_{nm}^{(1)}(N) + a_{nm}^{(2)}(N) + a_{nm}^{(3)}(N) = -\frac{\sum_j k_j^2 - k}{N(N-1)},$$

and the statement (9) is proved with  $\varkappa = \sum_j k_j^2 - k > 0$ .

**7. Proofs of theorems.** Our method is similar to the approach which was proposed in [4]. It is based on an embedded Markov chain  $\zeta_N(n, \omega) := x(\tau_n, \omega)$ ,  $n = 0, 1, \dots$ . The chain  $\zeta_N(n)$  is defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , evolves in discrete time and take its values in the set  $\mathbb{R}^N$ . A sequence  $\tau_1(\omega) \leq \tau_2(\omega) \leq \dots$  consists of time moments at which particles make jumps,  $\tau_0(\omega) \equiv 0$ . Evidently,  $\{\tau_{n+1} - \tau_n\}_{n=0}^\infty$  is a sequence of i.i.d random variables having exponential distribution with the mean value  $\gamma_N = (\alpha N + \delta)^{-1}$ . In the discrete Markov chain  $\zeta_N(n)$  one-step transitions have the following form: with probability  $\alpha \gamma_N$  particle  $i$  makes a jump  $x_i \rightarrow x_i + z$ , where  $z$  is distributed according to the law  $\rho(dz)$ ,  $i = \overline{1, N}$ , or with probability  $\delta \gamma_N$  a synchronization take place, namely, a collection of indices  $(i_1, \dots, i_k)$  is chosen randomly and then the particle configuration  $x$  is transformed into  $J_{k_1, \dots, k_l}^{(i_1, \dots, i_k)} x$ . By the law of large numbers  $\tau_n \sim \gamma_N n$  as  $n \rightarrow \infty$ , therefore, an asymptotic behavior of the particle system  $x(t)$ ,  $t \geq 0$ , can be reduced to asymptotic properties of the embedded Markov chain  $\zeta_N(n)$ . We are interested in the following sequences:  $s(n) := \mathbb{E} M(x(\tau_n))$

and  $d(n) := \mathbb{E}V(x(\tau_n))$ . Straightforward calculations together with Lemmas 1 and 2 show that

$$\begin{aligned}\mathbb{E}(M(\zeta_N(n+1)) \mid x(t), t \leq \tau_n) &= M(\zeta_N(n)) + \gamma_N \alpha a, \\ \mathbb{E}(V(\zeta_N(n+1)) \mid x(t), t \leq \tau_n) &= V(\zeta_N(n)) + \gamma_N \left( \alpha b_2 - \delta \frac{\varkappa}{N(N-1)} V(\zeta_N(n)) \right).\end{aligned}$$

Taking expected values of the both parts in the above equations we come to recurrent relations

$$s(n+1) = s(n) + \gamma_N \alpha a, \quad (10)$$

$$d(n+1) = d(n) \left( 1 - \gamma_N \delta \frac{\varkappa}{N(N-1)} \right) + \gamma_N \alpha b_2 \quad (11)$$

From (10) the statement of Theorem 1 easily follows. By iterating the equation (11), we get

$$d(n) = d(0) \left( 1 - \gamma_N \delta \frac{\varkappa}{N(N-1)} \right)^n + (\gamma_N \alpha b_2) \sum_{j=1}^{n-1} \left( 1 - \gamma_N \delta \frac{\varkappa}{N(N-1)} \right)^j. \quad (12)$$

Substituting  $n = \gamma_N^{-1} t(N)$  in (12) and letting  $N$  go to the infinity, we come to the conclusion of Theorem 2.

Theorem 3 can be derived from (12) by a straightforward calculation performed separately for each time stage.

**8. Possible generalizations and perspectives.** Undoubtedly the results obtained in this paper remain also true, if the synchronization  $L_s$  is taken in the form of a symmetrized polynomial interaction of any order, i.e., in the case when  $L_s$  is a finite sum (taken over index  $k \geq 2$ ) of  $k$ -particle symmetrized interactions. Apparently, the methods of our paper can be adapted also for other classes of the free evolutions  $L_0$ , such as independent Brownian motions.

The stochastic particle system considered here plays a basic role for future study of asymptotic behavior of general many-component stochastic systems with synchronization. The statement of Theorem 2 put forward a hypothesis that, seemingly, the result of the paper [4] on the existence of three phases of collective behavior remains true for a wide class of large particle systems with synchronization and, hence, it is not really caused by the specific nature of the model [4]. As it is seen from the present paper, key elements in the future proofs of such results should be some analogues of Lemma 2. On this way we expect to have difficulties with anisotropic synchronizations which are interesting in a number of important applications. Let us remark that in papers [2, 7], where some examples of anisotropic interactions were considered, the behavior of particle systems was studied only on a so called hydrodynamic scale while in the present paper we consider all possible time scales  $t(N)$ .

Unfortunately, an important class of cascade synchronizations [5, 6] can not be considered in the framework of the present approach.

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